

On the geometry of generalized Gaussian distributions *

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May 11, 2007

Abstract

In this paper we consider the space of those probability distributions which maximize the q -Rényi entropy. These distributions have the same parameter space for every q , and in the $q = 1$ case these are the normal distributions. Some methods to endow this parameter space with Riemannian metric is presented: the second derivative of the q -Rényi entropy, Tsallis-entropy and the relative entropy give rise to a Riemannian metric, the Fisher-information matrix is a natural Riemannian metric, and there are some geometrically motivated metrics which were studied by Siegel, Calvo and Oller, Lovrić, Min-Oo and Ruh. These metrics are different therefore our differential geometrical calculations based on a unified metric, which covers all the above mentioned metrics among others. We also compute the geometrical properties of this metric, the equation of the geodesic line with some special solutions, the Riemann and Ricci curvature tensors and scalar curvature. Using the correspondence between the volume of the geodesic ball and the scalar curvature we show how the parameter q modulates the statistical distinguishability of close points. We show that some frequently used metric in quantum information geometry can be easily recovered from classical metrics.

1 Introduction

In theoretical statistics and in applications the distance functions between probability distributions play an important role. The construction of a proper distance function has been considered by several authors. But even the same statistical model with different mathematical frameworks can lead to different distance functions. To narrow the family of potential distance functions we consider those which are natural from differential geometrical point of view.

Historically the pioneering work of Mahalanobis [23] was generalized by Rao [30], who first suggested the idea of considering the Fisher information [14] as a Riemannian metric on the space of probability distributions. Cencov [8] was the first to study monotone metrics on statistical manifolds. He proved that, up to a normalization, there exists a unique monotone metric, the Fisher information. Amari [3] and Amari and Nagaoka [4] provide modern account of the general differential geometry that arises from the Fisher information metric. The Fisher metric was studied further by Akin [1], James [16], Burbea [6], Mitchell [22], Atkinson and Mitchell [5], Skovgaard [34], Oller [25], Oller and Cuadrada [27], Oller and Corcuera [26] among other researchers. The combination of differential geometrical and statistical studies helped to find the statistical interpretation of geometrical quantities. For example the geodesic distance between probability distributions, which is usually known as Rao distance is a natural distance function between probability distributions; the statistical meaning of the so-called

*keywords: Gaussian distribution, differential geometry; MSC: 94A17, 53B21

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e-curvature was first clarified by Efron [12]; the normalized volume measure of the manifold is called Jeffreys' prior [17] within the field of Bayesian statistics.

In this paper we consider the space of those probability distributions which maximize the q -Rényi entropy. These distributions have the same parameter space for every q , and in the $q = 1$ case these are the normal distributions. The first results about the geometrical properties of these spaces are due to Amari [3, 2]. He considered the Fisher information metric on these manifolds and computed some geometrical invariants. Some methods to endow the parameter space with Riemannian metric is presented: the second derivative of the q -Rényi entropy [31], Tsallis-entropy [35] and the relative entropy give rise to a Riemannian metric, the Fisher-information matrix is a natural Riemannian metric, and there are some geometrically motivated metrics which were studied by Siegel [33], Calvo and Oller [7] and Lovrić, Min-Oo and Ruh [32]. These metrics are different therefore our differential geometrical calculations based on a unified metric, which covers all the above mentioned metrics among others. We also compute the geometrical properties of this metric, the equation of the geodesic line with some special solutions, the Riemann and Ricci curvature tensors and scalar curvature. Using the correspondence between the volume of the geodesic ball and the scalar curvature we show how the parameter q modulates the statistical distinguishability of close points. We show that some frequently used metric in quantum information geometry can be easily recovered from classical metrics.

2 q -Rényi entropy maximizing distributions

The normal distributions can be introduced as a result of the maximum entropy principle. Consider the family of density functions which are continuous and supported on the real line with given expectation value $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}$. Introducing the Lagrange multipliers a, b, c we have the following functional on the family of probability distributions

$$S(p) = - \int p(x) \log p(x) \, dx - a \left(\int p(x) \, dx - 1 \right) - b \left(\int p(x)x \, dx - \mu \right) - c \left(\int p(x)(x - \mu)^2 \, dx - \sigma^2 \right).$$

The variation of the functional is

$$\delta S = \int (-\log p(x) - 1 - a - bx - c(x - \mu)^2) \delta p(x) \, dx.$$

The functional has extremal point at p if its variation is zero. One can show that the entropy functional has local maximum at the point

$$p(x) = \exp(-a - bx - c(x - \mu)^2)$$

for appropriate parameters $a, b, c \in \mathbb{R}$.

The family of one dimensional normal distributions S_1 can be parameterized by the expectation value $u \in \mathbb{R}$ and the parameter $d \in \mathbb{R}^+$ as

$$f(d, u, x) = \frac{\sqrt{d}}{\sqrt{2\pi}} e^{-\frac{1}{2}d(x-u)^2}.$$

This means that S_1 can be identified with a 2 dimensional space $\Xi_1 = \mathbb{R}^+ \times \mathbb{R}$. The statistical properties of the distributions lead us to define Riemannian metric on the space Ξ_1 .

In general, the family of n dimensional normal distributions S_n can be parameterized by the expectation vector $\underline{u} \in \mathbb{R}^n$ and the inverse of the covariance matrix D . Let us denote the set of real

symmetric strictly positive $n \times n$ matrices by \mathcal{M}_n . Then we can identify the sets S_n and $\Xi_n = \mathcal{M}_n \times \mathbb{R}^n$ using the following one-to-one map

$$\Xi_n \rightarrow S_n \quad (D, \underline{u}) \mapsto f(D, \underline{u}, \cdot),$$

where

$$f(D, \underline{u}, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R} \quad \underline{x} \mapsto \frac{\sqrt{\det D}}{\sqrt{(2\pi)^n}} \exp\left(-\frac{1}{2}\langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle\right)$$

Normal distributions with zero expectation will said to be to special normal distributions. The parameter space of the n dimensional special normal distribution is $\Xi_n^{(s)} = \mathcal{M}_n$.

One can generalize the above mentioned procedure to extend the notion of Gaussian distributions using the q -Rényi entropy [31]. Let us fix a parameter $q \in \mathbb{R}^+ \setminus \{1\}$ and consider a density function p . The q -Rényi entropy of the distribution p is

$$S_q(p) = \frac{1}{1-q} \log \int_{\mathbb{R}} p(x)^q \, dx$$

if the integral exists.

The q -Rényi entropy maximizing distribution is the following. For a given $n \in \mathbb{N} \setminus \{0\}$ the parameter space is $\Xi_n = \mathcal{M}_n \times \mathbb{R}^n$. For a parameter $(D, \underline{u}) \in \Xi$ define the set

$$\text{Dom}(p, D, \underline{u}) = \begin{cases} \mathbb{R}^n, & \text{if } p \in \left] \frac{n}{n+2}, 1 \right[; \\ \left\{ \underline{x} \in \mathbb{R}^n \mid 1 + \frac{1-p}{2p-n(1-p)} \langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle \geq 0 \right\}, & \text{if } p > 1; \end{cases}$$

and define the density function as

$$f_p(D, \underline{u}, \underline{x}) = \begin{cases} A_{n,p} \sqrt{\det D} \left(1 + \frac{1-p}{2p-n(1-p)} \langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle \right)^{\frac{1}{p-1}}, & \text{if } \underline{x} \in \text{Dom}(p, D, \underline{u}); \\ 0, & \text{if } \underline{x} \notin \text{Dom}(p, D, \underline{u}). \end{cases}$$

The normalization constant of the generalized p -Gaussian distribution is

$$A_{n,p} = \begin{cases} \left(\frac{1-p}{2p-n(1-p)} \right)^{\frac{n}{2}} \frac{\Gamma\left(\frac{1}{1-p}\right)}{\pi^{\frac{n}{2}} \Gamma\left(\frac{1}{1-p} - \frac{n}{2}\right)}, & \text{if } p \in \left] \frac{n}{n+2}, 1 \right[; \\ \left(\frac{p-1}{2p-n(1-p)} \right)^{\frac{n}{2}} \frac{\Gamma\left(\frac{p}{p-1} + \frac{n}{2}\right)}{\pi^{\frac{n}{2}} \Gamma\left(\frac{p}{p-1}\right)}, & \text{if } p > 1. \end{cases}$$

For a given parameters $n \in \mathbb{N} \setminus \{0\}$ and $p > \frac{n}{n+2}$ we call

$$M_p = \{f_p(D, \underline{u}, \cdot) \mid (D, \underline{u}) \in \Xi_n\}$$

the family of p -generalized Gaussian distributions. This forms a manifold parameterized by (D, \underline{u}) . This is an α -family of probability distributions, where $\alpha = 2p-1$ and is α -flat (see Amari and Nagaoka [4]). The present paper studies the geometrical structures of M_p .

If we consider the limit $q \rightarrow 1$ then the q -Rényi entropy tends to the entropy. From this point on we will allow the $p = 1$ case, and we will consider it as the usual Gaussian distribution, and in the $p = 1$ case we sometimes omit the index p . The set

$$\mathcal{N} = \left\{ (n, p) \in \mathbb{N} \times \mathbb{R} \mid n > 0, p > \frac{n}{n+2} \right\}$$

can be considered as the label set of the p -Gaussian distributions, and for every pair $(n, p) \in \mathcal{N}$ the parameter space of the n -dimensional p -Gaussian distributions is $\Xi_n = \mathcal{M}_n \times \mathbb{R}^n$ and the parameter space of the special Gaussian distributions is $\Xi_n^{(s)} = \mathcal{M}_n$.

We present a Theorem which shows the maximum q -Rényi entropy property of the p -Gaussian distributions in the $q = p$ case. The maximum Rényi entropy problem was solved by Moriguti in the scalar case [24]. The distribution function was remarked by Zografos [37] in the multivariate case, but not connected to the Rényi entropy. The problem was solved first by Kapur [19] in the multivariate case, Johnson and Vignat also solved the problem in the multivariate case [18] using the result of Lutwak, Yang and Zhang [21]. Costa, Hero and Vignat [9] established properties of multivariate distributions maximizing Rényi-entropy, under a covariance constraint.

Theorem 2.1. *For any probability density $g : \mathbb{R}^n \rightarrow \mathbb{R}^+$ with fixed covariance matrix K , expectation $\underline{u} \in \mathbb{R}^n$ and parameter $q > \frac{n}{n+2}$,*

$$S_q(g) \leq S_q(f_q(K^{-1}, \underline{u}, \cdot)),$$

with equality if and only if $g = f_q(K^{-1}, \underline{u}, \cdot)$ almost everywhere.

Important to note, that the p -Gaussian distributions maximize not only the q -Rényi entropy, but the Tsallis entropy too, defined by equation (9) and minimize α -relative entropy (defined in the next Section) between the uniform distribution and an arbitrary one.

We call the family of probability distributions $f_p(D, \underline{u}, \cdot)$ $((n, p) \in \mathcal{N}, (D, \underline{u}) \in \Xi_n)$ extended Gaussian distributions.

3 Riemannian metrics on the space of extended Gaussian distributions

The parameter spaces Ξ_n and $\Xi_n^{(s)}$ have a natural manifold structure. Let us denote the space of real symmetric $n \times n$ matrices by M_n . Then at the point $(D, \underline{u}) \in \Xi_n$ the tangent space $T_{(D, \underline{u})} \Xi_n$ can be identified by $T_n = M_n \times \mathbb{R}^n$, since one can consider the tangent vector (X, \underline{x}) as a derivation defined for any smooth function $h : \Xi \rightarrow \mathbb{R}$ as

$$\frac{\partial h(D, \underline{u})}{\partial (X, \underline{x})} = \left. \frac{d}{dt} h(D + tX, \underline{u} + t\underline{x}) \right|_{t=0}. \quad (1)$$

In this setting a map

$$g : \Xi_n \times T_n \times T_n \rightarrow \mathbb{C} \quad ((D, \underline{u}), (X, \underline{x}), (Y, \underline{y})) \mapsto g_{D, \underline{u}}((X, \underline{x}), (Y, \underline{y}))$$

will be called a Riemannian metric if the following conditions hold. For all $(D, \underline{u}) \in \Xi_n$ the map

$$g_{D, \underline{u}} : T_n \times T_n \rightarrow \mathbb{C} \quad ((X, \underline{x}), (Y, \underline{y})) \mapsto g_{D, \underline{u}}((X, \underline{x}), (Y, \underline{y}))$$

is a scalar product and for all $(X, \underline{x}) \in T_n$ the map

$$g((X, \underline{x}), (X, \underline{x})) : \Xi_n \rightarrow \mathbb{C} \quad (D, \underline{u}) \mapsto g_{D, \underline{u}}((X, \underline{x}), (X, \underline{x}))$$

is smooth.

Now we present some ideas how the space Ξ_n can be endowed with Riemannian metric. For example the $(q$ -Rényi) entropy can generate a Riemannian metric: because the following Theorem shows that q -Rényi entropy is a convex functional, so its second derivative is a strictly positive symmetric linear map, and therefore it can define a Riemannian metric.

Theorem 3.1. For every pair $(n, p) \in \mathcal{N}$ and $(D, \underline{x}) \in \Xi_n$ the q -Rényi entropy ($q \in \mathbb{R}^+$) of the distribution $f_p(D, \underline{u}, \cdot)$ is

if $p, q > 1$:

$$S_q(f_p(D, \underline{u}, \cdot)) = \frac{n}{2} \log \frac{\pi(2p - n(1 - p))}{p - 1} + \frac{1}{1 - q} \log \left[\frac{\Gamma\left(\frac{p}{p-1} + \frac{n}{2}\right)^q \Gamma\left(\frac{q}{p-1} + 1\right)}{\Gamma\left(\frac{p}{p-1}\right)^q \Gamma\left(\frac{q}{p-1} + 1 + \frac{n}{2}\right)} \right] - \frac{1}{2} \log \det D \quad (2)$$

if $p < 1$, $q > \frac{n(1-p)}{2}$:

$$S_q(f_p(D, \underline{u}, \cdot)) = \frac{n}{2} \log \frac{\pi(2p - n(1 - p))}{1 - p} + \frac{1}{1 - q} \log \left[\frac{\Gamma\left(\frac{1}{1-p}\right)^q \Gamma\left(\frac{q}{1-p} - \frac{n}{2}\right)}{\Gamma\left(\frac{1}{1-p} - \frac{n}{2}\right)^q \Gamma\left(\frac{q}{1-p}\right)} \right] - \frac{1}{2} \log \det D. \quad (3)$$

Proof. First we compute the integral

$$I = \int_{\text{Dom}(p, D, \underline{u})} f_p(D, \underline{u}, \underline{x})^q \, d\underline{x}. \quad (4)$$

Choose our new coordinate system in \mathbb{R}^n parallel to the eigenvectors of D . In this coordinate system D is diagonal, with entries $(\lambda_i)_{i=1, \dots, n}$. If $p > 1$ then with the variables $a = \frac{p-1}{2p-n(p-1)}$ and $y_i = \sqrt{a\lambda_i}(x_i - u_i)$ the integral is

$$I = \frac{A_{n,p}^q (\det D)^{\frac{q-1}{2}}}{a^{\frac{n}{2}}} \int_{B_n(1)} \left(1 - \sum_{k=1}^n y_k^2\right)^{\frac{q}{p-1}} \, d\underline{y}.$$

In spherical coordinates this equation is

$$I = \frac{A_{n,p}^q (\det D)^{\frac{q-1}{2}}}{a^{\frac{n}{2}}} \int_0^1 (1 - r^2)^{\frac{q}{p-1}} r^{n-1} F_n \, dr,$$

where F_n is the surface of the n dimensional sphere with unit radius

$$F_n = \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

Evaluating the integral

$$\int_0^1 (1 - r^2)^{\frac{q}{p-1}} r^{n-1} \, dr = \frac{\Gamma\left(\frac{q}{p-1} + 1\right) \Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{q}{p-1} + 1 + \frac{n}{2}\right)}$$

we have

$$I = \left(\frac{a}{\pi}\right)^{\frac{n(q-1)}{2}} \left(\frac{\Gamma\left(\frac{p}{p-1} + \frac{n}{2}\right)}{\Gamma\left(\frac{p}{p-1}\right)}\right)^q \frac{\Gamma\left(\frac{q}{p-1} + 1\right)}{\Gamma\left(\frac{q}{p-1} + 1 + \frac{n}{2}\right)} (\det D)^{\frac{q-1}{2}} \quad (5)$$

and this verifies the Equation (2). If $p < 1$ then the integral (4) is

$$I = \frac{A_{n,p}^q (\det D)^{\frac{q-1}{2}}}{a^{\frac{n}{2}}} \int_0^1 (1 + r^2)^{\frac{q}{p-1}} r^{n-1} F_n \, dr$$

after the substitutions $a = \frac{1-p}{2p-n(p-1)}$ and $y_i = \sqrt{a\lambda_i}(x_i - u_i)$. Evaluating the integral we get

$$I = \left(\frac{a}{\pi}\right)^{\frac{n(q-1)}{2}} \left(\frac{\Gamma\left(\frac{1}{1-p}\right)}{\Gamma\left(\frac{1}{1-p} - \frac{n}{2}\right)}\right)^q \frac{\Gamma\left(\frac{q}{1-p} - \frac{n}{2}\right)}{\Gamma\left(\frac{q}{1-p}\right)} (\det D)^{\frac{q-1}{2}} \quad (6)$$

which leads to Equation (3). \square

Since the q -Rényi entropy is independent of the expectation vector \underline{u} the entropy cannot generate a Riemannian metric on the whole space Ξ just on $\Xi_n^{(s)}$. The q -Rényi entropy can be written in the form of

$$S_q(f_p(D, \underline{u}, \cdot)) = C_{n,p,q} - \frac{1}{2} \log \det D, \quad (7)$$

so the quadratic form generated by the functional S_q on the space of p -Gaussian distribution for every point $D \in \Xi_n^{(s)}$ and tangent vectors $X, Y \in T_n$ being

$$g_D^{(R)}(X, Y) = \frac{\partial^2}{\partial s \partial t} S_p(f_q(D + tX + sY, \underline{0}, \cdot)) \Big|_{t=s=0} = -\frac{1}{2} \frac{\partial^2}{\partial s \partial t} (\log \det(D + tX + sY)) \Big|_{t=s=0}$$

is independent of q and p .

Theorem 3.2. *For every pair $(n, p) \in \mathcal{N}$ for every point $D \in \Xi_n^{(s)}$ and for every tangent vectors $X, Y \in T_n$ we have*

$$g_D^{(R)}(X, Y) = \frac{1}{2} \text{Tr}(D^{-1} X D^{-1} Y) \quad (8)$$

for the quadratic form generated by the q -Rényi entropy.

Proof. To compute the derivative of the function $-\frac{1}{2} \log \det D$ we use the following equalities for symmetric strictly positive matrices

$$\log \det A = \text{Tr} \log A \quad \log A = \int_0^\infty (E + \tau E)^{-1} - (A + \tau E)^{-1} \, d\tau,$$

where E denotes the identity matrix. Then the derivative is

$$\begin{aligned} \frac{\partial^2}{\partial s \partial t} - \frac{1}{2} \log \det(D + tX + sY) \Big|_{t=s=0} &= -\frac{1}{2} \frac{\partial^2}{\partial s \partial t} \log(\det D)(\det(E + tX D^{-1} + sY D^{-1})) \Big|_{t=s=0} \\ &= -\frac{1}{2} \frac{\partial^2}{\partial s \partial t} \log \det(E + tX D^{-1} + sY D^{-1}) \Big|_{t=s=0} = -\frac{1}{2} \frac{\partial^2}{\partial s \partial t} \text{Tr} \log(E + tX D^{-1} + sY D^{-1}) \Big|_{t=s=0} \\ &= -\frac{1}{2} \text{Tr} \int_0^\infty \frac{\partial^2}{\partial s \partial t} ((E + \tau E)^{-1} - (E + tX D^{-1} + sY D^{-1} + \tau E)^{-1}) \Big|_{t=s=0} \, d\tau \\ &= \frac{1}{2} \text{Tr} \int_0^\infty (E + \tau E)^{-1} (Y D^{-1} (E + \tau E)^{-1} X D^{-1} + X D^{-1} (E + \tau E)^{-1} Y D^{-1}) (E + \tau E)^{-1} \, d\tau \\ &= \left(\text{Tr} X D^{-1} Y D^{-1} \right) \int_0^\infty (1 + \tau)^{-3} \, d\tau. \end{aligned}$$

This proves the equality $g_D^{(R)}(X, Y) = \frac{1}{2} \text{Tr}(D^{-1} X D^{-1} Y)$. \square

The Tsallis entropy [35] of the probability distribution f is defined as

$$S^{(q)}(f) = \frac{1}{1-q} \left(\int_{\mathbb{R}} f(x)^q \, dx - 1 \right) \quad (9)$$

for parameter $q \in \mathbb{R}^+ \setminus \{1\}$. Let us denote the quadratic form generated by the Tsallis entropy by $g^{(T,p,q)}$, i.e. for every point $D \in \Xi_n^{(s)}$ and tangent vectors $X, Y \in T_n$

$$g_D^{(T,p,q)}(X, Y) = \frac{\partial^2}{\partial s \partial t} S^{(q)}(f_p(D + tX + sY, \underline{0}, \cdot)) \Big|_{t=s=0}$$

if $S^{(q)}(f_p(D + tX + sY, \underline{0}, \cdot))$ is well defined.

Theorem 3.3. *For every pair $(n, p) \in \mathcal{N}$ for every point $D \in \Xi_n^{(s)}$ and for every tangent vectors $X, Y \in T_n$ we have the following expressions for the quadratic form generated by the entropy*

$$g_D^{(T,p,1)}(X, Y) = g_D^{(T,1,1)}(X, Y) = \frac{1}{2} \text{Tr}(D^{-1} X D^{-1} Y). \quad (10)$$

If $S^{(q)}(f_p(D, \underline{u}, \cdot))$ is well defined, then the generated quadratic form is

$$g_D^{(T,p,q)}(X, Y) = A'_{n,p,q} \det(D)^{\frac{q-1}{2}} \left(\text{Tr}(D^{-1} X D^{-1} Y) - \frac{q-1}{2} \text{Tr} X D^{-1} \text{Tr} Y D^{-1} \right), \quad (11)$$

where

$$A'_{n,p,q} = \begin{cases} \frac{1}{2} \left(\frac{p-1}{\pi(2p-n(p-1))} \right)^{\frac{n(q-1)}{2}} \times \left(\frac{\Gamma\left(\frac{p}{p-1} + \frac{n}{2}\right)}{\Gamma\left(\frac{p}{p-1}\right)} \right)^q \frac{\Gamma\left(\frac{q}{p-1} + 1\right)}{\Gamma\left(\frac{q}{p-1} + 1 + \frac{n}{2}\right)}, & \text{if } p > 1, q > 0; \\ \frac{1}{2} \left(\frac{1-p}{\pi(2p-n(p-1))} \right)^{\frac{n(q-1)}{2}} \times \left(\frac{\Gamma\left(\frac{1}{1-p}\right)}{\Gamma\left(\frac{1}{1-p} - \frac{n}{2}\right)} \right)^q \frac{\Gamma\left(\frac{q}{1-p} - \frac{n}{2}\right)}{\Gamma\left(\frac{q}{1-p}\right)}, & \text{if } p < 1, q > \frac{n(1-p)}{2}. \end{cases}$$

Proof. Since we have the limit

$$\lim_{q \rightarrow 1} S^{(q)}(f_p(D, \underline{u}, \cdot)) = \lim_{q \rightarrow 1} S_q(f_p(D, \underline{u}, \cdot)) = S(f_p(D, \underline{u}, \cdot))$$

the formula $g_D^{(T,1,q)}(X, Y) = g_D^{(T,1,1)}(X, Y)$ is straightforward from the Equation (7) and the metric was computed in the previous Theorem.

Now let us compute the derivative of the determinant function.

$$\begin{aligned} \frac{d}{dt}(\det(D + tX)) \Big|_{t=0} &= \det D \frac{d}{dt} \det(E + tX D^{-1}) \Big|_{t=0} = \det D \frac{d}{dt}(\exp \text{Tr} \log(D + tX)) \Big|_{t=0} \\ &= \det D \text{Tr} \frac{d}{dt} \int_0^\infty (E + \tau E)^{-1} - (E + tX D^{-1} + \tau E)^{-1} \Big|_{t=0} d\tau \\ &= \det D \text{Tr} \int_0^\infty (E + \tau E)^{-1} X D^{-1} (E + \tau E)^{-1} d\tau = (\det D)(\text{Tr} X D^{-1}). \end{aligned}$$

This can be expressed as $(d \det)(D)(X) = (\det D)(\text{Tr} X D^{-1})$. The second derivative of the determinant is

$$(d^2 \det)(D)(X)(Y) = (\det D)(\text{Tr} X D^{-1})(\text{Tr} Y D^{-1}) - (\det D)(\text{Tr} X D^{-1} Y D^{-1}).$$

According to these equalities the derivative of the function $(\det D)^{\frac{p-1}{2}}$ is

$$\begin{aligned} (d^2 \det^{\frac{p-1}{2}})(D)(X)(Y) &= \frac{\partial^2}{\partial s \partial t} (\det(D + tX + sY))^{\frac{p-1}{2}} \Big|_{t=s=0} \\ &= \frac{p-1}{2} (\det D)^{\frac{p-1}{2}} \left(-\text{Tr} X D^{-1} Y D^{-1} + \frac{p-1}{2} \text{Tr} X D^{-1} \text{Tr} Y D^{-1} \right). \end{aligned}$$

This gives the Equation (11) and the constants $A'_{n,p,q}$ come from the Equations (5,6). \square

The Fisher information matrix is defined on parametric probability distributions. At a point $(D, \underline{u}) \in \Xi_n$ the quadratic form

$$g_{D, \underline{u}}^{(F, q)}((X, \underline{x}), (Y, \underline{y})) = \int_{\mathbb{R}^n} f_q(D, \underline{u}, \underline{z}) \frac{\partial \log f_q(D, \underline{u}, \underline{z})}{\partial (X, \underline{x})} \frac{\partial \log f_q(D, \underline{u}, \underline{z})}{\partial (Y, \underline{y})} d\underline{z}$$

(if the integral exists) gives rise to a positive definite matrix, so inducing a Riemannian metric on Ξ_n . This metric is often called the expected information metric for the family of probability density functions; the original ideas are due to Fisher [14] and Rao [30].

Theorem 3.4. *For every pair $(n, p) \in \mathcal{N}$, where $p < 2$ for every point $(D, \underline{u}) \in \Xi_n$ and for every tangent vectors $(X, \underline{x}), (Y, \underline{y}) \in T_n$ the Fisher information matrix of M_p is*

$$\begin{aligned} g_{D, \underline{u}}^{(F, p)}((X, \underline{x}), (Y, \underline{y})) &= \frac{1}{2(2-p)} \text{Tr}(D^{-1} X D^{-1} Y) + \frac{p-1}{4(2-p)} \text{Tr}(D^{-1} X) \text{Tr}(D^{-1} Y) \\ &\quad + \frac{2+n(p-1)}{(2p+n(p-1))(2-p)} \langle \underline{x}, D \underline{y} \rangle. \end{aligned}$$

Proof. At a point $(D, \underline{u}) \in \Xi$ we have

$$\log f_p(D, \underline{u}, \underline{x}) = \log A_{n,p} + \frac{1}{2} \log \det D + \frac{1}{p-1} \log \left(1 + \frac{1-p}{2p-n(1-p)} \langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle \right). \quad (12)$$

Choose our new coordinate system in \mathbb{R}^n parallel to the eigenvectors of D . In this coordinate system D is diagonal, with entries $(\lambda_i)_{i=1, \dots, n}$. Let us denote by $(\underline{e}_k)_{k=1, \dots, n}$ the orthonormal basis. According to Equation (1) the partial derivative of Equation (12) with respect to a basis vector is

$$\frac{\partial \log f_p(D, \underline{u}, \underline{x})}{\partial (0, \underline{e}_k)} = \frac{2}{2p-n(1-p)} \frac{1}{1 + \frac{1-p}{2p-n(1-p)} \langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle} \lambda_k (x_k - u_k). \quad (13)$$

First we consider the $p \in \left] \frac{n}{n+2}, 1 \right[$ case. The Fisher information is

$$\begin{aligned} g_{D, \underline{u}}^{(F, p)}((0, \underline{e}_k), (0, \underline{e}_l)) &= \\ \frac{4\lambda_k \lambda_l A_{n,p} \sqrt{\det D}}{(2p-n(1-p))^2} \int_{\mathbb{R}^n} \left(1 + \frac{1-p}{2p-n(1-p)} \langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle \right)^{\frac{1}{p-1}-2} (x_k - u_k)(x_l - u_l) d\underline{x} \end{aligned}$$

Introducing the new variables $a = \frac{1-p}{2p-n(1-p)}$, $y_i = \sqrt{a\lambda_i}(x_i - u_i)$ we have

$$g_{D, \underline{u}}^{(F, p)}((0, \underline{e}_k), (0, \underline{e}_l)) = \delta_{kl} \frac{A_{n,p}}{a^{\frac{n}{2}+1}} \frac{4\lambda_k}{(2p-n(1-p))^2} \int_{\mathbb{R}^n} \left(1 + \sum_{i=1}^n y_i^2 \right)^{\frac{1}{p-1}-2} y_k^2 d\underline{y}.$$

If $n = 1$

$$g_{D, \underline{u}}^{(F, p)}((0, \underline{e}_1), (0, \underline{e}_1)) = \frac{A_{1,p}}{\sqrt{a}a} \frac{4\lambda_1}{(3p-1)^2} \int_{-\infty}^{\infty} (1+y^2)^{\frac{1}{p-1}-2} y^2 dy = \lambda_1 \frac{1+p}{(3p-1)(2-p)}$$

and if $n > 1$ then in the spherical coordinates in $n-1$ dimension we have

$$g_{D, \underline{u}}^{(F, p)}((0, \underline{e}_k), (0, \underline{e}_l)) = \delta_{kl} \frac{A_{n,p}}{a^{\frac{n}{2}+1}} \frac{4\lambda_k}{(2p-n(1-p))^2} \int_0^\infty \int_{-\infty}^\infty (1+y_k^2+r^2)^{\frac{1}{p-1}-2} y_k^2 r^{n-2} F_{n-1} dy_k dr.$$

Using the integral formulas

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty (1 + y_k^2 + r^2)^{\frac{1}{p-1}-2} y_k^2 r^{n-2} \, dy_k \, dr &= \frac{\sqrt{\pi} \Gamma\left(\frac{1}{1-p} + \frac{1}{2}\right)}{2\Gamma\left(\frac{1}{1-p} + 2\right)} \int_0^\infty (1 + r^2)^{\frac{1}{p-1}-\frac{1}{2}} r^{n-2} \, dr \\ &= \frac{\Gamma\left(\frac{1}{1-p} + 1 - \frac{n}{2}\right) \Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{1}{1-p} + \frac{1}{2}\right)} \end{aligned}$$

and after some simplification we have

$$g_{D,\underline{u}}^{(F,p)}((0,\underline{x}), (0,\underline{y})) = \frac{2 + n(p-1)}{(2p + n(p-1))(2-p)} \langle \underline{x}, D\underline{y} \rangle \quad (14)$$

which is valid for every $n \in \mathbb{N} \setminus \{0\}$. Since D is diagonal the partial derivative of the Equation (12) is

$$\frac{\partial \log f_p(D, \underline{u}, \underline{x})}{\partial (E_{ii}, 0)} = \frac{1}{2\lambda_i} + \frac{a}{p-1} \frac{(x_i - u_i)^2}{1 + a\langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle}. \quad (15)$$

The Fisher information is

$$\begin{aligned} g_{D,\underline{u}}^{(F,p)}((E_{ii}, 0), (E_{kk}, 0)) &= \frac{1}{4} \frac{1}{\lambda_i \lambda_k} + \frac{a A_{n,p} \sqrt{\det D}}{2(p-1)\lambda_k} \int_{\mathbb{R}^n} (1 + a\langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle)^{\frac{1}{p-1}-1} (x_i - u_i)^2 \, d\underline{x} \\ &\quad + \frac{a A_{n,p} \sqrt{\det D}}{2(p-1)\lambda_i} \int_{\mathbb{R}^n} (1 + a\langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle)^{\frac{1}{p-1}-1} (x_k - u_k)^2 \, d\underline{x} \\ &\quad + \frac{a^2 A_{n,p} \sqrt{\det D}}{(p-1)^2} \int_{\mathbb{R}^n} (1 + a\langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle)^{\frac{1}{p-1}-2} (x_i - u_i)^2 (x_k - u_k)^2 \, d\underline{x}. \end{aligned}$$

The first integral is

$$\frac{A_{n,p}}{2(p-1)a^{\frac{n}{2}}\lambda_k\lambda_i} \int_0^\infty \int_{-\infty}^\infty (1 + y_i^2 + r^2)^{\frac{1}{p-1}-1} y_i^2 r^{n-2} F_{n-1} \, dy_i \, dr = -\frac{1}{4\lambda_i\lambda_k}.$$

The third one is if $i \neq k$

$$\frac{A_{n,p}}{(p-1)^2 a^{\frac{n}{2}} \lambda_k \lambda_i} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty (1 + y_i^2 + y_k^2 + r^2)^{\frac{1}{p-1}-2} y_i^2 y_k^2 r^{n-3} F_{n-2} \, dy_i \, dy_k \, dr = \frac{1}{4(2-p)\lambda_i\lambda_k}$$

and if $i = k$

$$\frac{A_{n,p}}{(p-1)^2 a^{\frac{n}{2}} \lambda_k^2} \int_0^\infty \int_{-\infty}^\infty (1 + y_k^2 + r^2)^{\frac{1}{p-1}-2} y_k^4 r^{n-2} F_{n-1} \, dy_k \, dr = \frac{3}{4(2-p)\lambda_k^2}.$$

Combining these integrals

$$g_{D,\underline{u}}^{(F,p)}((E_{ii}, 0), (E_{kk}, 0)) = \frac{1}{4} \left(\frac{1 + 2\delta_{ik}}{2-p} - 1 \right) \frac{1}{\lambda_i \lambda_k}.$$

If D is not diagonal this can be expressed as

$$g_{D,\underline{u}}^{(F,p)}((X, 0), (Y, 0)) = \frac{1}{2(2-p)} \text{Tr}(D^{-1} X D^{-1} Y) + \frac{p-1}{4(2-p)} \text{Tr}(D^{-1} X) \text{Tr}(D^{-1} Y). \quad (16)$$

Finally the formulas (14,16) give us the metric since

$$g_{D,\underline{u}}^{(F,p)}((E_{ii}, 0), (0, \underline{e}_k)) = 0. \quad (17)$$

If $p > 1$ then the partial derivatives given by the Equations (13,15) are the same. The Fisher information is

$$g_{D,\underline{u}}^{(F,p)}(0, \underline{e}_k), (0, \underline{e}_l) = \frac{4\lambda_k \lambda_l a^2 A_{n,p} \sqrt{\det D}}{(p-1)^2} \int_{\text{Dom}(p, D, \underline{u})} (1 - a\langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle)^{\frac{1}{p-1}-2} (x_k - u_k)(x_l - u_l) \, d\underline{x},$$

where $a = \frac{p-1}{2p-n(1-p)}$. Introducing the new variables $y_i = \sqrt{a\lambda_i}(x_i - u_i)$ we have

$$g_{D,\underline{u}}^{(F,p)}((0, \underline{e}_k), (0, \underline{e}_l)) = \delta_{kl} \frac{A_{n,p}}{a^{\frac{n}{2}}} \frac{4a\lambda_k}{(p-1)^2} \int_{B_n(1)} \left(1 - \sum_{i=1}^n y_i^2\right)^{\frac{1}{p-1}-2} y_k^2 \, d\underline{y},$$

where $B_n(1)$ is the closed unit ball in \mathbb{R}^n with center origin. If $n = 1$

$$g_{D,\underline{u}}^{(F,p)}((0, \underline{e}_1), (0, \underline{e}_1)) = \frac{A_{1,p}}{\sqrt{a}} \frac{4a\lambda_1}{(p-1)^2} \int_{-1}^1 (1 - y^2)^{\frac{1}{p-1}-2} y^2 \, dy = \lambda_1 \frac{1+p}{(3p-1)(2-p)}$$

and if $n > 1$ then in the spherical coordinates in $n-1$ dimension we have

$$g_{D,\underline{u}}^{(F,p)}((0, \underline{e}_k), (0, \underline{e}_l)) = \delta_{kl} \frac{A_{n,p}}{a^{\frac{n}{2}}} \frac{4a\lambda_k}{(p-1)^2} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} (1 - y_k^2 - r^2)^{\frac{1}{p-1}-2} y_k^2 r^{n-2} F_{n-1} \, dy_k \, dr.$$

Evaluating the integral

$$\int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} (1 - y_k^2 - r^2)^{\frac{1}{p-1}-2} y_k^2 r^{n-2} \, dy_k \, dr = \frac{1}{4} \frac{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{1}{p-1} - 1\right)}{\Gamma\left(\frac{1}{p-1} + \frac{n}{2}\right)}$$

we get again Equation (14) for every n . The Riemannian product of the matrix units is

$$\begin{aligned} g_{D,\underline{u}}^{(F,p)}((E_{ii}, 0), (E_{kk}, 0)) &= \frac{1}{4} \frac{1}{\lambda_i \lambda_k} - \frac{a A_{n,p} \sqrt{\det D}}{2(p-1)\lambda_k} \int_{\text{Dom}(p, D, \underline{u})} (1 - a\langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle)^{\frac{1}{p-1}-1} (x_i - u_i)^2 \, d\underline{x} \\ &\quad - \frac{a A_{n,p} \sqrt{\det D}}{2(p-1)\lambda_i} \int_{\text{Dom}(p, D, \underline{u})} (1 - a\langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle)^{\frac{1}{p-1}-1} (x_k - u_k)^2 \, d\underline{x} \\ &\quad + \frac{a^2 A_{n,p} \sqrt{\det D}}{(p-1)^2} \int_{\text{Dom}(p, D, \underline{u})} (1 - a\langle \underline{x} - \underline{u}, D(\underline{x} - \underline{u}) \rangle)^{\frac{1}{p-1}-2} (x_i - u_i)^2 (x_k - u_k)^2 \, d\underline{x}. \end{aligned}$$

Introducing the variables $y_i = \sqrt{a\lambda_i}(x_i - u_i)$ the domain $\text{Dom}(p, D, \underline{u})$ will be transformed to $B_n(1)$. Evaluating the integrals we get Equation (16). Finally we note that Equation (17) is valid in this $p < 1$ setting too, and this completes the proof. \square

In the $p \geq 2$ case the Fisher information does not exist, since the integral which defines it divergent.

Relative entropies are special distance functions between probability measures and although there are several relative entropy functions, but most of them are special Csiszár φ -divergences [10, 11]. Assume that $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a strictly convex function, and $\varphi(1) = 0$. Then one can define the Csiszár φ -relative entropy as

$$H^{(\varphi,p)}(f_p(D_1, \underline{u}_1, \cdot), f_p(D_2, \underline{u}_2, \cdot)) = \int_{\mathbb{R}^n} f_p(D_1, \underline{u}_1, \underline{x}) \varphi \left(\frac{f_p(D_2, \underline{u}_2, \underline{x})}{f_p(D_1, \underline{u}_1, \underline{x})} \right) d\underline{x}.$$

For example the Kullback–Liebler [20], Hellinger [15] and α -relative entropies are given by the functions $\varphi(x) = -\log x$, $\varphi(x) = (1 - \sqrt{x})^2$ and $\varphi(x) = \frac{4}{1-\alpha^2} \left(1 - x^{\frac{1+\alpha}{2}}\right)$. We note that the α -relative entropy is strongly related to the Rényi [31] and to the Tsallis entropy [35, 36]. The quadratic form induced by the φ -divergence is

$$g_{D, \underline{u}}^{(\varphi,p)}((X, \underline{x}), (Y, \underline{y})) = \frac{\partial^2}{\partial s \partial t} H^{(\varphi)}(f_p(D, \underline{u}, \cdot), f_p(D + tX + sY, \underline{u} + t\underline{x} + s\underline{y}, \cdot)) \Big|_{t=s=0}.$$

Theorem 3.5. *Assume that $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a strictly convex function and $\varphi(1) = 0$. Then $g^{(\varphi,p)} = \varphi''(1)g^{(F,p)}$ on the manifold Ξ .*

Proof. The computation

$$\begin{aligned} g_{D, \underline{u}}^{(\varphi,p)}((X, \underline{x}), (Y, \underline{y})) &= \frac{\partial^2}{\partial s \partial t} H^{(\varphi)}(f_p(D, \underline{u}, \cdot), f_p(D + tX + sY, \underline{u} + t\underline{x} + s\underline{y}, \cdot)) \Big|_{t=s=0} \\ &= \int_{\mathbb{R}^n} f_p(D, \underline{u}, \underline{z}) \left[\frac{\partial^2}{\partial s \partial t} \varphi \left(\frac{f_p(D + tX + sY, \underline{u} + t\underline{x} + s\underline{y}, \underline{z})}{f_p(D, \underline{u}, \underline{z})} \right) \Big|_{t=s=0} \right] d\underline{z} \\ &= \int_{\mathbb{R}^n} \frac{\varphi''(1)}{f_p(D, \underline{u}, \underline{z})} \cdot \frac{d f_p(D + tX, \underline{u} + t\underline{x}, \underline{z})}{d t} \Big|_{t=0} \cdot \frac{d f_p(D + sY, \underline{u} + s\underline{y}, \underline{z})}{d s} \Big|_{s=0} d\underline{z} \\ &= \varphi''(1) \int_{\mathbb{R}^n} \frac{1}{f_p(D, \underline{u}, \underline{z})} \cdot \frac{\partial f_p(D, \underline{u}, \underline{z})}{\partial (X, \underline{x})} \cdot \frac{\partial f_p(D, \underline{u}, \underline{z})}{\partial (Y, \underline{y})} d\underline{z} \\ &= \varphi''(1) g_{D, \underline{u}}^{(F,p)}((X, \underline{x}), (Y, \underline{y})) \end{aligned}$$

verifies the Theorem. □

Calvo and Oller studied a different metric on the space Ξ_n [7]. Their starting point was the metric

$$g : P(n) \times M_n \times M_n \rightarrow \mathbb{R} \quad (D, X, Y) \mapsto \frac{1}{2} \text{Tr}(D^{-1} X D^{-1} Y),$$

where $P(n)$ denotes the set of $n \times n$ real, symmetric, positive definite matrices, and the embedding

$$\pi : \mathcal{M}_n \times \mathbb{R}^{n+1} \times \mathbb{R}^+ \rightarrow P(n) \quad (K, \underline{u}, \beta) \mapsto \begin{pmatrix} K + \beta \underline{u} \circ \underline{u} & \beta \underline{u} \\ \beta \underline{u} & \beta \end{pmatrix}.$$

The metric g has been studied by Siegel [33], James [16] and Burbea [6]. Calvo and Oller considered the pull-back metric of g by π restricted to the manifold $\mathcal{M}_n \times \mathbb{R}^n \times \{\beta\}$, which is

$$\tilde{g}_\pi|_{\mathcal{M}_n \times \mathbb{R}^n \times \{\beta\}} : (\mathcal{M}_n \times \mathbb{R}^n) \times T_n \times T_n \rightarrow \mathbb{R} \quad (K, \underline{u}), (X, \underline{x}), (Y, \underline{y}) \mapsto \frac{1}{2} \text{Tr}(K^{-1} X K^{-1} Y) + \beta \langle \underline{x}, K^{-1} \underline{y} \rangle.$$

If we use our parametrization of the normal distributions, namely the inverse of the covariance matrix and the expectation vector, then the metric is

$$g^{(\text{CO},\beta)} : \Xi_n \times \mathbb{T}_n \times \mathbb{T}_n \rightarrow \mathbb{R} \quad (D, \underline{u}), (X, \underline{x}), (Y, \underline{y}) \mapsto \frac{1}{2} \text{Tr}(D^{-1} X D^{-1} Y) + \beta \langle \underline{x}, D \underline{y} \rangle.$$

Lovrić, Min-Oo and Ruh [32] studied a slightly different metric on the space Ξ_n . Let us sketch their fundamental idea briefly. Denote by $P_1(n)$ the set of $n \times n$ real, symmetric, positive definite matrices with determinant 1. Then the map

$$j : \mathcal{M}_n \times \mathbb{R}^n \rightarrow P_1(n+1) \quad (K, \underline{u}) \mapsto (\det K)^{-\frac{2}{n+1}} \begin{pmatrix} K^2 + \underline{u} \circ \underline{u} & \underline{u} \\ \underline{u}^T & 1 \end{pmatrix}$$

is a smooth bijection. The special linear group has a natural smooth group action on $P_1(n)$

$$A : \text{SL}(n) \rightarrow \text{Aut}(P_1(n)) \quad g \mapsto \left(m \mapsto g m g^T \right),$$

where g^T denotes the transpose of g . This group action represents $P(n)$ as the Riemannian symmetric space $\text{SL}(n)/\text{SO}(n)$ with $\text{SO}(n)$ principal bundle

$$\text{SL}(n) \rightarrow P(n) \quad g \mapsto g g^T.$$

This means that the space $\mathcal{M}_n \times \mathbb{R}^n$ can be considered as a Riemannian symmetric space $\text{SL}(n+1)/\text{SO}(n+1)$. It is known in the theory of symmetric spaces, that the natural $\text{SL}(n+1)$ invariant metric on the space $P_1(n+1)$ is given by restricting the Killing form of the simple Lie algebra $\mathfrak{sl}(n+1)$ to the subspace $\mathfrak{so}(n+1)$ under the Cartan decomposition $\mathfrak{sl}(n+1) = \mathfrak{o}(n+1) \oplus \mathfrak{so}(n+1)$. The generated metric is unique up to a positive constant factor. This metric at a point $(K, \underline{u}) \in \mathcal{M}_n \times \mathbb{R}^n$ for tangent vectors $(X, \underline{x}), (Y, \underline{y}) \in \mathbb{T}_n$ is given by the equation

$$g_{K, \underline{u}}^{(\text{inv.})}((X, \underline{x}), (Y, \underline{y})) = \text{Tr}(K^{-1} X K^{-1} Y) - \frac{1}{n+1} (\text{Tr } K^{-1} X) (\text{Tr } K^{-1} Y) + \frac{1}{2} \langle \underline{x}, K^{-1} \underline{y} \rangle.$$

Using the inverse of the covariance matrix as a parameter, this metric is

$$g_{D, \underline{u}}^{(\text{LMR})}((X, \underline{x}), (Y, \underline{y})) = \text{Tr}(D^{-1} X D^{-1} Y) - \frac{1}{n+1} (\text{Tr } D^{-1} X) (\text{Tr } D^{-1} Y) + \frac{1}{2} \langle \underline{x}, D \underline{y} \rangle.$$

Corollary 3.1. *On the parameter space of the special normal distributions we have the equality of the metrics*

$$g^{(R)} = g^{(T,p,1)} = g^{(F,1)}|_{\Xi_n^{(s)}} = g^{(\text{CO},\beta)}|_{\Xi_n^{(s)}}.$$

On the parameter space of the normal distributions we have

$$g^{(F,1)} = g^{(\text{CO},1)},$$

but the metrics $g^{(F,p)}$, $g^{(\text{CO},\beta)}$ and $g^{(\text{LMR})}$ are pairwise incomparable in the sense that there is no $(n,p) \in \mathcal{N}$ parameter such that two of these metrics are equal up to a multiplicative factor.

To work with the Riemannian metrics $g^{(R)}$, $g^{(T,1,q)}$, $g^{(F,q)}$, $g^{(\varphi,q)}$, $g^{(\text{CO},\beta)}$ and $g^{(\text{LMR})}$ simultaneously we consider the metric

$$g_{D, \underline{u}}((X, \underline{x}), (Y, \underline{y})) = \frac{1}{2} \text{Tr}(D^{-1} X D^{-1} Y) + \alpha (\text{Tr } D^{-1} X) (\text{Tr } D^{-1} Y) + \beta \langle \underline{x}, D \underline{y} \rangle$$

with parameters $\alpha, \beta \in \mathbb{R}$, $\alpha \neq -\frac{1}{2n}$, $\beta \neq 0$ on the manifold Ξ_n . In the $\beta = 0$ case we restrict the manifold to $\Xi_n^{(s)}$. For every $D \in \Xi_n^{(s)}$ and $X, Y \in T_n$ we have a Cauchy-Schwarz inequality

$$\text{Tr}^2(D^{-1}XD^{-1}Y) \leq \text{Tr}(D^{-1}XD^{-1}X) \text{Tr}(D^{-1}YD^{-1}Y).$$

Substituting $Y = D$ we have

$$\frac{1}{2} \text{Tr}(D^{-1}XD^{-1}X) - \frac{1}{2n} (\text{Tr} D^{-1}X)^2 \geq 0.$$

It means that if $\alpha > -\frac{1}{2n}$ then g is a Riemannian metric, if $\alpha < -\frac{1}{2n}$ then g is a semi-Riemannian metric, and in the $\alpha = -\frac{1}{2n}$ case g is a degenerated quadratic form. The Theorems and proofs are valid for semi-Riemannian metrics too, so we have just one condition $\alpha \neq -\frac{1}{2n}$.

4 Geodesics

In this section we derive the differential equation of the geodesic lines in the space M_n and we present some solutions.

Theorem 4.1. *A curve $\gamma : \mathbb{R} \rightarrow \Xi_n$, $\gamma(t) = (D(t), \underline{u}(t))$ is a geodesic curve if and only if for every $t \in \text{Dom } \gamma$*

$$\begin{aligned} \ddot{D}(t) &= \dot{D}(t)D(t)^{-1}\dot{D}(t) + \beta(D(t)\underline{\dot{u}}(t)) \circ (D(t)\underline{\dot{u}}(t)) - \frac{2\alpha\beta}{1+2n\alpha} \langle \underline{\dot{u}}(t), D(t)\underline{\dot{u}}(t) \rangle D(t) \\ \underline{\ddot{u}}(t) &= -D(t)^{-1}\dot{D}(t)\underline{\dot{u}}(t) \end{aligned} \quad (18)$$

holds.

Proof. Denote by $\text{GL}(n)$ the set of invertible $n \times n$ matrices and define the reciprocal function as

$$i : \text{GL}(n) \rightarrow \text{GL}(n) \quad D \mapsto D^{-1}.$$

At the point D the tangent space $T_D \text{GL}(n)$ can be identified with the set of $n \times n$ matrices $\text{Mat}(n)$. The derivative of the inversion function is

$$di : \text{GL}(n) \rightarrow \text{Lin}(\text{Mat}(n), \text{Mat}(n)) \quad (D) \mapsto (di)(D) = \left(A \mapsto (di)(D)(A) = -D^{-1}AD^{-1} \right).$$

This leads to the derivative of the metric

$$\begin{aligned} dg : \Xi_n &\rightarrow \text{Lin}(T_n, \text{Lin}(T_n \times T_n, \mathbb{R})) \\ (D, \underline{u}) &\mapsto \left((Z, \underline{z}) \mapsto (((X, \underline{x}), (Y, \underline{y})) \mapsto dg_{D, \underline{u}}(Z, \underline{z})((X, \underline{x}), (Y, \underline{y}))) \right), \end{aligned}$$

where

$$\begin{aligned} dg_{D, \underline{u}}(Z, \underline{z})((X, \underline{x}), (Y, \underline{y})) &= -\frac{1}{2} \text{Tr} D^{-1}(ZD^{-1}X + XD^{-1}Z)D^{-1}Y + \beta \langle \underline{x}, D\underline{y} \rangle \\ &\quad - \alpha \text{Tr}(D^{-1}ZD^{-1}X) \text{Tr}(D^{-1}Y) - \alpha \text{Tr}(D^{-1}X) \text{Tr}(D^{-1}ZD^{-1}Y). \end{aligned}$$

At a given point $(D, \underline{u}) \in \Xi_n$ for given tangent vectors $(X, \underline{x}), (Y, \underline{y}) \in T_n$ the map

$$\begin{aligned} \tau_{(D, \underline{u}), (X, \underline{x}), (Y, \underline{y})} : T_n &\rightarrow \mathbb{R} \\ (Z, \underline{z}) &\mapsto \frac{1}{2} \left(dg_{D, \underline{u}}(Y, \underline{y})((X, \underline{x}), (Z, \underline{z})) + dg_{D, \underline{u}}(X, \underline{x})((Y, \underline{y}), (Z, \underline{z})) - dg_{D, \underline{u}}(Z, \underline{z})((X, \underline{x}), (Y, \underline{y})) \right) \end{aligned}$$

is a linear functional. It means that there exists a unique tangent vector $V_{(D,\underline{u}),(X,\underline{x}),(Y,\underline{y})} \in T_n$ such that for all vectors $(Z,\underline{z}) \in T_n$

$$g_{D,\underline{u}}(V_{(D,\underline{u}),(X,\underline{x}),(Y,\underline{y})}, (Z,\underline{z})) = \tau_{(D,\underline{u}),(X,\underline{x}),(Y,\underline{y})}(Z,\underline{z})$$

holds. One can define the map

$$\Gamma : \Xi_n \rightarrow \text{Lin}(T_n \times T_n, T_n) \quad (D,\underline{u}) \mapsto \left(((X,\underline{x}), (Y,\underline{y})) \mapsto V_{(D,\underline{u}),(X,\underline{x}),(Y,\underline{y})} \right)$$

which is called covariant derivative. It means, that the equation for all tangent vectors $(Z,\underline{z}) \in T_n$

$$\begin{aligned} g_{D,\underline{u}}(\Gamma_{(D,\underline{u})}(X,\underline{x})(Y,\underline{y}), (Z,\underline{z})) = & -\frac{1}{4} \text{Tr} \left(D^{-1}(XD^{-1}Y + YD^{-1}X)D^{-1}Z \right) \\ & -\frac{\alpha}{2} \text{Tr}(D^{-1}(XD^{-1}Y + YD^{-1}X)) \text{Tr}(D^{-1}Z) \\ & +\frac{\beta}{2} (\langle \underline{y}, X\underline{z} \rangle + \langle \underline{x}, Y\underline{z} \rangle - \langle \underline{x}, Z\underline{y} \rangle) \end{aligned} \quad (19)$$

determines the covariant derivative. Let us write the covariant derivative in the form of

$$\Gamma_{(D,\underline{u})}(X,\underline{x})(Y,\underline{y}) = \left(-\frac{1}{2}(XD^{-1}Y + YD^{-1}X) + DW, D^{-1}\underline{w} \right)$$

for some $W \in T_n$ and $\underline{w} \in \mathbb{R}^n$. Then Equation (19) is

$$\frac{1}{2} \text{Tr}(WD^{-1}Z) + \alpha \text{Tr} W \text{Tr} D^{-1}Z + \beta \langle \underline{w}, \underline{z} \rangle = \frac{\beta}{2} (\langle \underline{y}, X\underline{z} \rangle + \langle \underline{x}, Y\underline{z} \rangle - \langle \underline{x}, Z\underline{y} \rangle). \quad (20)$$

Let us introduce the notation \odot for symmetrized diadic product, for vectors $\underline{u}, \underline{v} \in \mathbb{R}^n$

$$\underline{u} \odot \underline{v} = \underline{u} \circ \underline{v} + \underline{v} \circ \underline{u},$$

that is, the components of the $n \times n$ matrix $(\underline{u} \odot \underline{v})$ are $(\underline{u} \odot \underline{v})_{ij} = \underline{u}_i \underline{v}_j + \underline{u}_j \underline{v}_i$. Equation (20) means that the vector component of the covariant derivative is

$$\underline{w} = \frac{1}{2}(X\underline{y} + Y\underline{x})$$

and the remaining matrix part is

$$W = -\frac{\beta}{2}(\underline{x} \odot \underline{y})D + \gamma E$$

where the parameter γ can easily be found. Combining the terms together we have the following expression for the covariant derivative.

$$\Gamma_{(D,\underline{u})}(X,\underline{x})(Y,\underline{y}) = \left(-\frac{1}{2}(XD^{-1}Y + YD^{-1}X) - \frac{\beta}{2}D(\underline{x} \odot \underline{y})D + \frac{2\alpha\beta}{1+2n\alpha} \langle \underline{x}, D\underline{y} \rangle D, \frac{1}{2}D^{-1}(X\underline{y} + Y\underline{x}) \right) \quad (21)$$

A curve $\gamma : \mathbb{R} \rightarrow \Xi$ is called a geodesic curve if

$$\forall t \in \text{Dom}(\gamma) : \quad \ddot{\gamma}(t) + \Gamma_{\gamma(t)}(\dot{\gamma}(t))(\dot{\gamma}(t)) = 0$$

holds. Consider a curve $\gamma : \mathbb{R} \rightarrow \Xi$, $\gamma(t) = (D(t), \underline{u}(t))$, substitute it into the equation of the geodesic curve and expand the covariant derivative, then according to Equation (21) we get Equation (18) of the Theorem. \square

We have some remarks about the geodesic curves, which are only valid for Riemannian metrics.

Remark 4.1. Let us consider the case $n = 1$, and assume that $\alpha \neq -\frac{1}{2n}$. Then the system of differential equations of the geodesic line is

$$\ddot{D}(t) = \frac{\dot{D}(t)^2}{D(t)} + \frac{\beta}{1+2\alpha} D(t)^2 \dot{u}(t)^2 \quad \ddot{u}(t) = -\frac{\dot{D}(t)}{D(t)} \dot{u}(t).$$

The curve $\gamma : \mathbb{R} \rightarrow \Xi_1$

$$\gamma(t) = \left(\frac{2}{a^2} \cosh^2(bt + c), a \sqrt{\frac{1+2\alpha}{\beta}} \tanh(ct + b) + d \right)$$

is a geodesic line.

Assume that we have two points $(D_0, \underline{u}_0), (D_1, \underline{u}_1)$ in the space Ξ_1 and assume that $\underline{u}_1 > \underline{u}_0$. Let us define the following quantities

$$\begin{aligned} x &= (\underline{u}_1 - \underline{u}_0) \sqrt{\frac{D_0 \beta}{2 + 4\alpha}} & y &= \sqrt{\frac{D_1}{D_0}} & c &= \log \left(\frac{\sqrt{(x^2 y^2 + 1 - y^2)^2 + 4x^2 y^4} - (x^2 y^2 + 1 - y^2)}{2xy^2} \right) \\ a &= \sqrt{\frac{2}{D_0}} \cosh c & b &= -\log(y(1 - x e^c)) & d &= \underline{u}_0 - a \sqrt{\frac{1+2\alpha}{\beta}} \tanh c. \end{aligned}$$

Then the curve $\gamma : [0, 1] \rightarrow \Xi_1$

$$\gamma(t) = \left(\frac{2}{a^2} \cosh^2(bt + c), a \sqrt{\frac{1+2\alpha}{\beta}} \tanh(bt + c) + d \right)$$

is a geodesic line, such that $\gamma(0) = (\underline{u}_0, D_0)$ and $\gamma(1) = (\underline{u}_1, D_1)$. Simple calculation shows that the distance between the points $(\underline{u}_0, D_0), (\underline{u}_1, D_1) \in \Xi_1$ is

$$d((\underline{u}_0, D_0), (\underline{u}_1, D_1)) = \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt = \sqrt{2 + 4\alpha} |b|.$$

The geodesic line and the Rao distance on the space of special normal distributions has been computed by Siegel [33] and Burbea [6]. The next Remark concerns their results.

Remark 4.2. Let us consider the space of n dimensional special normal distributions $\Xi_n^{(s)}$ and assume that $\alpha \neq -\frac{1}{2n}$. Then the curve $D : \mathbb{R} \rightarrow \Xi_n^{(s)}$ is a geodesic line if and only if

$$\ddot{D}(t) = \dot{D}(t) D(t)^{-1} \dot{D}(t).$$

Assume that we have two points D_0, D_1 in the space $\Xi_n^{(s)}$. Then the curve

$$\gamma : [0, 1] \rightarrow \Xi \quad t \mapsto D_0^{\frac{1}{2}} \exp \left(t \log \left(D_0^{-\frac{1}{2}} D_1 D_0^{-\frac{1}{2}} \right) \right) D_0^{\frac{1}{2}} \quad (22)$$

is a geodesic line, such that $\gamma(0) = D_0$ and $\gamma(1) = D_1$. The distance between the points $D_0, D_1 \in \Xi_n^{(s)}$ is

$$d(D_0, D_1) = \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt = \sqrt{\frac{1}{2} \text{Tr} \log^2 \left(D_0^{-\frac{1}{2}} D_1 D_0^{-\frac{1}{2}} \right) + \alpha \text{Tr}^2 \log \left(D_0^{-\frac{1}{2}} D_1 D_0^{-\frac{1}{2}} \right)}.$$

Remark 4.3. In the $\alpha = 0$ case the metric is the pull-back of the Siegel metric by the embedding

$$\pi_\beta : \Xi_n \rightarrow P(n+1) \quad (D, \underline{u}) \mapsto \begin{pmatrix} D^{-1} + \beta \underline{u} \odot \underline{u} & \beta \underline{u} \\ \beta \underline{u} & \beta \end{pmatrix}.$$

The equation of the geodesic line in the Siegel metric is given by Equation (22). If we have two points $(D_0, \underline{u}_0), (D_1, \underline{u}_1) \in \Xi_n$ then we define the matrices $S_i = \pi_\beta(D_i, \underline{u}_i)$ ($i = 0, 1$), the equation of the geodesic line is

$$\gamma : [0, 1] \rightarrow \Xi \quad t \mapsto \pi_\beta^{-1} \left[S_0^{\frac{1}{2}} \exp \left(t \log \left(S_0^{-\frac{1}{2}} S_1 S_0^{-\frac{1}{2}} \right) \right) S_0^{\frac{1}{2}} \right]$$

and the distance between the points is

$$d((D_0, \underline{u}_0), (D_1, \underline{u}_1)) = \sqrt{\frac{1}{2} \text{Tr} \log^2 \left(S_0^{-\frac{1}{2}} S_1 S_0^{-\frac{1}{2}} \right)}.$$

Remark 4.4. In the $\alpha = 0$ case we can more exact parametrization of the geodesic line in some special cases. Assume that $B, C \in \mathcal{M}_n$ are diagonal matrices, $\underline{A}, \underline{D} \in \mathbb{R}^n$ are vectors such that the components of \underline{A} are equal and U is an $n \times n$ orthogonal matrix such that $U \underline{A} = \underline{A}$. Then the curve

$$\gamma : \mathbb{R}^+ \rightarrow \Xi \quad t \mapsto \left(\frac{2}{\|\underline{A}\|^2} U \cosh^2(Bt + C) U^{-1}, \sqrt{\frac{n}{\beta}} U \tanh(Bt + C) \underline{A} + \underline{D} \right)$$

is a geodesic line. The distance between the points $\gamma(t_0)$ and $\gamma(t_1)$ ($t_0, t_1 \in \mathbb{R}^+$) is

$$d(\gamma(t_0), \gamma(t_1)) = \int_{t_0}^{t_1} \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt = |t_1 - t_0| \sqrt{2 \text{Tr} B^2}.$$

5 Curvatures

Since Efron clarified the statistical meaning of the curvature, different curvature tensors has been studied on statistical manifolds. For curvatures on the space of normal distributions see for example Amari [2, 3], Siegel [33], Burbea [6], Skovgaard [34], Calvo and Oller [7], Lovrić Min-Oo and Ruh [32].

Theorem 5.1. *For every point $(D, \underline{u}) \in \Xi_n$ and for every tangent vectors $(X, \underline{x}), (Y, \underline{y}), (Z, \underline{z}) \in T_n$ the Riemann curvature tensor is*

$$\begin{aligned} R_{(D, \underline{u})}((X, \underline{x}), (Y, \underline{y}), (Z, \underline{z})) &= \left[\frac{1}{4} \left(Z D^{-1} X D^{-1} Y + Y D^{-1} X D^{-1} Z - X D^{-1} Y D^{-1} Z - Z D^{-1} Y D^{-1} X \right) \right. \\ &+ \frac{\beta}{4} \left(Y \underline{x} \odot D \underline{z} - X \underline{y} \odot D \underline{z} + D \underline{y} \odot Z \underline{x} - D \underline{x} \odot Z \underline{y} \right) + \frac{\alpha \beta}{1 + 2n\alpha} \left(\langle X \underline{y} - Y \underline{x}, \underline{z} \rangle + \langle \underline{x}, Z \underline{y} \rangle - \langle \underline{y}, Z \underline{x} \rangle \right) D, \\ &\frac{1}{4} \left(D^{-1} (Y D^{-1} X - X D^{-1} Y) \underline{z} + D^{-1} Z D^{-1} (X \underline{y} - Y \underline{x}) \right) + \frac{\beta}{4} \left(\underline{z} \odot \underline{x} D \underline{y} - (\underline{z} \odot \underline{y}) D \underline{x} \right) \\ &\left. + \frac{\alpha \beta}{1 + 2n\alpha} \left(\langle \underline{y}, D \underline{z} \rangle \underline{x} - \langle \underline{x}, D \underline{z} \rangle \underline{y} \right) \right]. \end{aligned} \quad (23)$$

Proof. The derivative of the covariant derivative is

$$\begin{aligned} d\Gamma : \Xi_n &\rightarrow \text{Lin}(T_n, \text{Lin}(T_n \times T_n, T_n)) \\ (D, \underline{u}) &\mapsto \left((X, \underline{x}) \mapsto ((Y, \underline{y}), (Z, \underline{z})) \mapsto d\Gamma_{(D, \underline{u})}(X, \underline{x})(Y, \underline{y})(Z, \underline{z}) \right) \end{aligned}$$

where from Equation (21)

$$\begin{aligned} d\Gamma_{(D,\underline{u})}(Z,\underline{z})(X,\underline{x})(Y,\underline{y}) &= \frac{1}{2}(XD^{-1}ZD^{-1}Y + YD^{-1}ZD^{-1}X) - \frac{\beta}{2}\left(Z(\underline{x} \odot \underline{y})D + D(\underline{x} \odot \underline{y})Z\right) \\ &\quad + \frac{2\alpha\beta}{1+2n\alpha}\left(\langle \underline{x}, Z\underline{y} \rangle D + \langle \underline{x}, D\underline{y} \rangle Z\right) - \frac{1}{2}D^{-1}ZD^{-1}(X\underline{y} + Y\underline{x}). \end{aligned}$$

The Riemann curvature tensor is defined to be

$$R : \Xi_n \rightarrow \text{Lin}(\mathbb{T}_n \times \mathbb{T}_n \times \mathbb{T}_n, \mathbb{T}_n) \quad (D, \underline{u}) \mapsto \left(((X, \underline{x})(Y, \underline{y})(Z, \underline{z})) \mapsto R_{(D, \underline{u})}((X, \underline{x}), (Y, \underline{y}), (Z, \underline{z})) \right),$$

where

$$\begin{aligned} R_{(D, \underline{u})}((X, \underline{x}), (Y, \underline{y}), (Z, \underline{z})) &= d\Gamma_{(D, \underline{u})}(X, \underline{x})(Y, \underline{y})(Z, \underline{z}) - d\Gamma_{(D, \underline{u})}(Y, \underline{y})(X, \underline{x})(Z, \underline{z}) \\ &\quad + \Gamma_{(D, \underline{u})}((X, \underline{x}), \Gamma_{(D, \underline{u})}(Y, \underline{y})(Z, \underline{z})) - \Gamma_{(D, \underline{u})}((Y, \underline{y}), \Gamma_{(D, \underline{u})}(X, \underline{x})(Z, \underline{z})). \end{aligned}$$

We omit the details of the straightforward, but lengthy calculation of the curvature tensor. \square

Theorem 5.2. *For every point $(D, \underline{u}) \in \Xi_n$ and for every tangent vectors $(X, \underline{x}), (Y, \underline{y}) \in \mathbb{T}_n$ the Ricci curvature tensor is*

$$\text{Ric}_{(D, \underline{u})}((X, \underline{x}), (Y, \underline{y})) = -\frac{n+1}{4} \text{Tr}(D^{-1}XD^{-1}Y) + \frac{1}{4} \text{Tr}(D^{-1}X) \text{Tr}(D^{-1}Y) - \frac{\beta}{2(1+2n\alpha)} \langle \underline{x}, D\underline{y} \rangle. \quad (24)$$

Proof. At a point $(D, \underline{u}) \in \Xi_n$ for given tangent vectors $(X, \underline{x}), (Y, \underline{y}) \in \mathbb{T}_n$ the map

$$R_{(D, \underline{u})}(\cdot, (X, \underline{x}), (Y, \underline{y})) : \mathbb{T}_n \rightarrow \mathbb{T}_n \quad (Z, \underline{z}) \mapsto R_{(D, \underline{u})}((Z, \underline{z}), (X, \underline{x}), (Y, \underline{y}))$$

is linear, and its trace is the Ricci tensor

$$\text{Ric} : \Xi_n \rightarrow \text{Lin}(\mathbb{T}_n \times \mathbb{T}_n, \mathbb{R}) \quad (D, \underline{u}) \mapsto \left(((X, \underline{x}), (Y, \underline{y})) \mapsto \text{Ric}_{(D, \underline{u})}((X, \underline{x}), (Y, \underline{y})) \right),$$

where

$$\text{Ric}_{(D, \underline{u})}((X, \underline{x}), (Y, \underline{y})) = \text{Tr} R_{(D, \underline{u})}(\cdot, (X, \underline{x}), (Y, \underline{y})).$$

The elements $\text{Ric}_{(D, \underline{u})}((X, \underline{x}), (X, \underline{x}))$ determines the Ricci tensor. For the further calculation we fix the tangent vector $(X, \underline{x}) \in \mathbb{T}_n$. According to the Equation (23) the Riemann curvature tensor consists of six summands. We compute the trace of the summands separately. Let us denote by E_{ij} the usual system of $n \times n$ matrix unit and define

$$F_{ij} = E_{ij} + E_{ji}$$

for indices $1 \leq i < j \leq n$. To compute the trace we choose the basis

$$\{E_{ii}\}_{i=1, \dots, n} \cup \{F_{ij}\}_{1 \leq i < j \leq n} \cup \{e_i\}_{i=1, \dots, n} \quad (25)$$

in \mathbb{T}_n , where $(e_i)_{i=1, \dots, n}$ is the canonical basis in \mathbb{R}^n . The trace of the first summand is

$$\begin{aligned} \rho_1 &= \frac{1}{2} \sum_{i=1}^n \text{Tr} (XD^{-1}E_{ii}D^{-1}XE_{ii} - E_{ii}D^{-1}XD^{-1}XE_{ii}) \\ &\quad + \frac{1}{4} \sum_{1 \leq i < j \leq n} \text{Tr} (XD^{-1}F_{ij}D^{-1}XF_{ij} - F_{ij}D^{-1}XD^{-1}XF_{ij}) \end{aligned}$$

which is

$$\rho_1 = -\frac{n+1}{4} \text{Tr } D^{-1} X D^{-1} X + \frac{1}{4} \text{Tr } X^2 D^{-2} + \frac{1}{4} \text{Tr}^2 D^{-1} X.$$

The trace of the second summand is

$$\rho_2 = -\frac{\beta}{4} \sum_{i=1}^n \text{Tr}(E_{ii}(E_{ii}\underline{x} \odot D\underline{x})) - \frac{\beta}{8} \sum_{1 \leq i < j \leq n} \text{Tr } F_{ij}(\underline{x} \odot D\underline{x}) F_{ij} = -\beta \frac{n+1}{4} \langle \underline{x}, D\underline{x} \rangle.$$

The third summand gives

$$\rho_3 = \frac{\alpha\beta}{1+2n\alpha} \sum_{i=1}^n \text{Tr} \langle E_{ii}\underline{x}, \underline{x} \rangle D E_{ii} + \frac{\alpha\beta}{1+2n\alpha} \sum_{1 \leq i < j \leq n} \text{Tr} \langle F_{ij}\underline{x}, \underline{x} \rangle D F_{ij} = \frac{\alpha\beta}{1+2n\alpha} \langle \underline{x}, D\underline{x} \rangle.$$

The trace of the forth, fifth and sixth summand is

$$\begin{aligned} \rho_4 &= \frac{1}{4} \sum_{i=1}^n \langle e_i, -D^{-1} X D^{-1} X e_i \rangle = -\frac{1}{4} \text{Tr } D^{-1} X D^{-1} X \\ \rho_5 &= \frac{\beta}{4} \sum_{i=1}^n \langle e_i, (\underline{x} \odot e_i) D\underline{x} - (\underline{x} \odot \underline{x}) D e_i \rangle = \beta \frac{n-1}{4} \langle \underline{x}, D\underline{x} \rangle \\ \rho_6 &= \frac{\alpha\beta}{1+2n\alpha} \sum_{i=1}^n \langle e_i, \langle \underline{x}, D\underline{x} \rangle e_i - \langle e_i, D\underline{x} \rangle \underline{x} \rangle = \frac{\alpha\beta(n-1)}{1+2n\alpha} \langle \underline{x}, D\underline{x} \rangle. \end{aligned}$$

Adding the traces we have the diagonal element of the Ricci tensor

$$\text{Ric}_{(D, \underline{u})}((X, \underline{x}), (X, \underline{x})) = \sum_{i=1}^6 \rho_i = -\frac{n+1}{4} \text{Tr } D^{-1} X D^{-1} X + \frac{1}{4} \text{Tr}^2 D^{-1} X - \frac{\beta}{1+2n\alpha} \langle \underline{x}, D\underline{x} \rangle.$$

Using the polarization formula

$$\text{Ric}_{(D, \underline{u})}((X, \underline{x}), (Y, \underline{y})) = \frac{1}{4} \left(\text{Ric}_{(D, \underline{u})}((X+Y, \underline{x}+\underline{y}), (X+Y, \underline{x}+\underline{y})) - \text{Ric}_{(D, \underline{u})}((X-Y, \underline{x}-\underline{y}), (X-Y, \underline{x}-\underline{y})) \right)$$

we get Equation (24). \square

The next Theorem shows that the manifolds Ξ_n and $\Xi_n^{(s)}$ has constant scalar curvature.

Theorem 5.3. *For every point $D \in \Xi_n^{(s)}$ the scalar curvature of the space of special normal distributions is*

$$\text{Scal}_s(D) = -\frac{n(2(n-1)(n+1)(n+2)\alpha + n^2 + 2n - 1)}{4(1+2n\alpha)} \quad (26)$$

and for every point $(D, \underline{u}) \in \Xi_n$ the space of normal distributions is

$$\text{Scal}(D, \underline{u}) = -\frac{n(n+1)(2(n+2)(n-1)\alpha + n + 1)}{4(1+2n\alpha)}. \quad (27)$$

Proof. At a point $(D, \underline{u}) \in \Xi_n$ for given tangent vector $(X, \underline{x}) \in T_n$ the map

$$T_n \rightarrow \mathbb{R} \quad (Y, \underline{y}) \mapsto \text{Ric}_{(D, \underline{u})}((X, \underline{x}), (Y, \underline{y}))$$

defines a linear functional. So there exists a unique $(\tilde{X}, \tilde{\underline{x}}) \in T_n$ tangent vector, such that

$$g_{(D, \underline{u})}((\tilde{X}, \tilde{\underline{x}}), (Y, \underline{y})) = \text{Ric}_{(D, \underline{u})}((X, \underline{x}), (Y, \underline{y}))$$

holds for every tangent vector $(Y, \underline{y}) \in T_n$. Let us define the map

$$\widetilde{\text{Ric}} : \Xi_n \rightarrow \text{Lin}(T_n, T_n) \quad (D, \underline{u}) \mapsto \left((X, \underline{x}) \mapsto (\tilde{X}, \tilde{\underline{x}}) \right).$$

The explicit expression

$$\widetilde{\text{Ric}}_{(D, \underline{u})}(X, \underline{x}) = -\frac{n+1}{2}X + \frac{1+2(n+1)\alpha}{2(1+2n\alpha)}D \text{Tr}(D^{-1}X) - \frac{1}{2(1+2n\alpha)}\underline{x} \quad (28)$$

can be easily verified. The scalar curvature of the manifold is the trace of the map $\widetilde{\text{Ric}}$

$$\text{Scal} : \Xi_n \rightarrow \mathbb{R} \quad (D, \underline{u}) \mapsto \text{Tr} \widetilde{\text{Ric}}_{D, \underline{u}}.$$

Using the basis (25) the trace of the three summand in the Equation (28) is

$$\begin{aligned} \rho'_1 &= -\frac{n+1}{2} \sum_{i=1}^n \text{Tr}(E_{ii}E_{ii}) - \frac{n+1}{4} \sum_{1 \leq i < j \leq n} \text{Tr}(F_{ij}F_{ij}) = -\frac{n(n+1)^2}{4} \\ \rho'_2 &= \frac{1+2(n+1)\alpha}{2(1+2n\alpha)} \sum_{i=1}^n \text{Tr}(E_{ii}D) \text{Tr}(E_{ii}D^{-1}) + \frac{1+2(n+1)\alpha}{4(1+2n\alpha)} \sum_{1 \leq i < j \leq n} \text{Tr}(F_{ij}D) \text{Tr}(F_{ij}D^{-1}) \\ &= n \frac{1+2(n+1)\alpha}{4(1+2n\alpha)} \\ \rho'_3 &= -\frac{1}{2(1+2n\alpha)} \sum_{i=1}^n \langle e_i, e_i \rangle = -\frac{n}{2(1+2n\alpha)}. \end{aligned}$$

The scalar curvature of the manifold Ξ_n at a point $(D, \underline{u}) \in \Xi_n$ is

$$\text{Scal}(D, \underline{u}) = \rho'_1 + \rho'_2 + \rho'_3$$

and the scalar curvature of the space of special normal distributions at a point $D \in \Xi_n^{(s)}$ is

$$\text{Scal}_s(D, \underline{u}) = \rho'_1 + \rho'_2.$$

□

6 Conclusion

Finally we have some remarks about the geometry of the generalized Gaussian distributions.

Remark 6.1. For every pair $(n, p) \in \mathcal{N}$, where $p < 2$ the scalar curvature of the space of extended Gaussian distribution endowed with the Fisher information metric at every point is

$$\text{Scal} = -\frac{n(n+1)(2-p)}{4(2+n(p-1))}((n+2)(n-1)(p-1) + 2(n+1)).$$

We note that the parameter p is in the interval $\left] \frac{n}{n+2}, 2 \right[$, it means that the scalar curvature is a monotonously increasing function of p . The scalar curvature at a given point is connected to the

statistical distinguishability of the point from its neighborhood, since the first nonconstant term in the Taylor expansion of the volume of the geodesic ball is the scalar curvature

$$V_n(r) = \frac{r^n \pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \left(1 - \frac{\text{Scal}}{6(n+2)} r^2 + O(r^4) \right).$$

This idea is widely used in quantum information geometry and in that framework it is due to Petz [29]. In this classical setting this means, that the parameter p modulates the statistical properties of this manifold. Namely, in the $p \rightarrow 2$ limit the manifold is more homogenous and it is more difficult to distinguish close points in the $p \rightarrow \frac{n}{n+2}$ limit; it is easier to decide whether two points are identical or just close to each other. This can have relevance in hypothesis testing.

Remark 6.2. Consider the space of special normal distributions and the Fisher information matrix

$$g_D(X, Y) = \text{Tr}(D^{-1} X D^{-1} Y).$$

Surprisingly from this well-known classical metric one can easily recover some metrics which are frequently used in quantum information theory. In quantum setting just the trace one matrices of $\Xi_n^{(s)}$ are considered. For example the Riemannian metrics

$$g_D^{(\text{KM})}(X, Y) = \int_0^\infty g_{D+tE, \underline{u}}(X, Y) \, dt$$

$$g_D^{(\text{La})}(X, Y) = g_D(D^{1/2} X, Y D^{1/2})$$

are very important ones in quantum setting, they are called Kubo–Mori [13, 28] metric and largest metric. This kind of differential geometrical connections can help to understand and to interpret the geometrical invariants of the quantum information manifolds.

Acknowledgement. This work was supported by Japan Society for the Promotion of Science, contract number P 06917.

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